

Symmetries of asymptotically flat axisymmetric spacetimes with null dust

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Symmetries of spacetimes with null dust field as a source compatible with asymptotic flatness are studied by using the Bondi-Sachs-van der Burg formalism. It is shown that in an axially symmetric spacetime with null dust field in which at least locally a smooth null infinity in the sense of Penrose exists, the only allowable additional Killing vector forming with the axial one a two-dimensional Lie algebra (the axial and the additional Killing vector are not assumed to be hypersurface orthogonal) is a supertranslational Killing vector and the gravitational field is then non-radiative (the Weyl tensor has a non-radiative character).

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I. INTRODUCTION AND SUMMARY

Recently a unique role of boost-rotation symmetric electrovacuum spacetimes describing “uniformly accelerated particles” of various kinds [1] was exhibited by a theorem which states that in axially symmetric, asymptotically flat spacetimes the only additional symmetry that does not exclude radiation is the boost symmetry (in Ref. [2] for vacuum spacetimes with hypersurface orthogonal Killing vectors and in Ref. [3] for electrovacuum spacetimes with Killing vectors which are in general not hypersurface orthogonal). Our effort in this paper is to prove a similar theorem for asymptotically flat spacetimes with null dust fields. We also specialize the spacetime to be axially symmetric (with the axial Killing vector which is in general not hypersurface orthogonal) - this assumption simplifies lengthy calculations.

If one is interested in gravitational radiation from a general bounded matter source, i.e., in the behaviour of gravitational field far from the source, one has to turn to approximation methods. The Bondi-Sachs formalism is a powerful instrument for the treatment not only of asymptotically flat vacuum [4,5] and electrovacuum [6] fields, but also for the investigation of asymptotically flat null dust fields [7]. The energy-momentum tensor of null dust or pure radiation

$$T_{\alpha\beta} = \rho n_\alpha n_\beta, \quad n_\alpha n^\alpha = 0, \quad \rho > 0, \quad (1)$$

with ρ being the energy density of the radiation field, describes a field of massless radiation propagating along a null congruence with the tangent vector n^α . This field is the incoherent superposition of waves with random phases and different polarizations where the radiation arises from electromagnetic null field, massless scalar field, neutrino field or gravitational field itself. The field equations of these originating fields are not considered. Exact solutions of this class are the Vaidya solution [8] which can model the exterior of a spherically symmetric shining star and the Kinnersley photon rocket [9], a particle emitting photons and accelerating because of the recoil. Also null dust fields with rotation are known [10,11].

In Sec. II we start out from the general form of an axially symmetric metric in Bondi-Sachs coordinates $\{u, r, \theta, \phi\}$, where the null coordinate u and the spherical angles θ, ϕ are constant along null rays while the luminosity distance r

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varies. We consider their asymptotic series expansions at $r \rightarrow \infty$ assuming the Einstein equations for the null dust field to be satisfied and the spacetime to be asymptotically flat.

Then we assume that, in addition, another symmetry exists, i.e., an additional Killing vector field which forms with the axial one a two-dimensional Lie algebra. By first decomposing the additional Killing vector field η^α in the null tetrad and then solving the Killing equations asymptotically in the leading order for this new Killing vector, we find that the additional Killing vector asymptotically generates either a supertranslation or a boost along the symmetry axis. However, developing and solving Killing equations in higher orders and considering Lie derivatives of the energy momentum tensor for null dust to vanish for both the supertranslational Killing vector in Sec. III and the boost Killing vector in Sec. IV we conclude that in fact the only allowable additional Killing vector of axially symmetric spacetimes with null dust is a supertranslational Killing vector and then the gravitational field is non-radiative (the Weyl tensor has a non-radiative character).

Our conventions for the Riemann and Ricci tensors follow those of Ref. [12] but our signature is -2 .

II. AXISYMMETRIC NULL DUST SPACETIMES WITH ANOTHER SYMMETRY

Consider an axially symmetric spacetime with the corresponding Killing vector field denoted by $\partial/\partial\phi$. Assume that at least the "piece of \mathcal{J}^+ " exists in the sense of Ref. [13]. Then one can introduce the Bondi-Sachs coordinate system $\{u, r, \theta, \phi\} \equiv \{x^0, x^1, x^2, x^3\}$ in which the metric has the form [4–6]

$$\begin{aligned} ds^2 = & \left(\frac{V}{r} e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 U W \sinh 2\delta \right) du^2 \\ & + 2e^{2\beta} du dr + 2r^2 (e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta) du d\theta + 2r^2 (e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta) \sin \theta du d\phi \\ & - r^2 [\cosh 2\delta (e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2) + 2 \sinh 2\delta \sin \theta d\theta d\phi] , \end{aligned} \quad (2)$$

where the six metric functions $U, V, W, \beta, \gamma, \delta$ do not depend on ϕ because of axial symmetry. The spacetime (2) is filled with null dust described by the energy-momentum tensor (1) which is assumed to be axially symmetric, too.

In Ref. [7] the expansions of the metric functions of (2) for asymptotically flat spacetimes with null dust in Bondi-Sachs coordinates are derived. For that, the null expansion vector n^α is chosen to be identified with the null vector k^α of the Bondi-Sachs tetrad (10) at \mathcal{J}^+ , i.e., \mathcal{J}^+ has to exist in the direction determined by the null vector n^α of the null dust. Then its contravariant components are

$$\begin{aligned} n^u &= \frac{\mathcal{U}(u, \theta)}{r^2} + \mathcal{O}(r^{-3}) , \\ n^r &= 1 + \frac{\mathcal{R}(u, \theta)}{r} + \mathcal{O}(r^{-2}) , \\ n^\theta &= \frac{\mathcal{T}(u, \theta)}{r^2} + \mathcal{O}(r^{-3}) , \\ n^\phi &= \frac{\mathcal{F}(u, \theta)}{r^2} + \mathcal{O}(r^{-3}) , \end{aligned} \quad (3)$$

and the covariant components read

$$\begin{aligned} n_u &= 1 + \frac{\mathcal{R}}{r} + \mathcal{O}(r^{-2}) , \\ n_r &= \frac{\mathcal{U}}{r^2} + \mathcal{O}(r^{-3}) , \\ n_\theta &= -\mathcal{T} + \mathcal{O}(r^{-1}) , \\ n_\phi &= -\mathcal{F} \sin^2 \theta + \mathcal{O}(r^{-1}) . \end{aligned} \quad (4)$$

As the vector n^α is null ($n^\alpha n_\alpha = 0$), functions entering (3), (4) have to satisfy

$$\mathcal{U} = \frac{1}{2}(\mathcal{T}^2 + \mathcal{F}^2 \sin^2 \theta) , \quad (5)$$

and similarly for the higher-order coefficients. Then the equations for the null dust field (1) can be solved. The metric coefficients have in the first order in r^{-k} the same form as Eq. (4) in Ref. [3]:

$$\begin{aligned}
\gamma &= \frac{c}{r} + \mathcal{O}(r^{-3}) , \\
\delta &= \frac{d}{r} + \mathcal{O}(r^{-3}) , \\
\beta &= -\frac{1}{4}(c^2 + d^2)\frac{1}{r^2} + \mathcal{O}(r^{-4}) , \\
U &= -(c_{,\theta} + 2c \cot \theta)\frac{1}{r^2} + \mathcal{O}(r^{-3}) , \\
W &= -(d_{,\theta} + 2d \cot \theta)\frac{1}{r^2} + \mathcal{O}(r^{-3}) , \\
V &= r - 2M + \mathcal{O}(r^{-1}) .
\end{aligned} \tag{6}$$

For the radiation density $\rho(u, r, \theta)$ we write

$$\rho(u, r, \theta) = \frac{\rho_2(u, \theta)}{r^2} + \mathcal{O}(r^{-3}) \tag{7}$$

and from the field equations

$$M_{,u} = -(c_{,u}^2 + d_{,u}^2) - \frac{1}{2}\kappa_0\rho_2 + \frac{1}{2}(c_{,\theta\theta} + 3c_{,\theta} \cot \theta - 2c)_{,u} \tag{8}$$

follows. The energy balance at null infinity (where null infinity admits a regular spherical cross section) then shows that the mass loss $m_{,u}$ results from a linear superposition of the pure and the gravitational radiation parts [7]

$$m_{,u} = -\frac{1}{2} \int_0^\pi (c_{,u}^2 + d_{,u}^2 + \frac{1}{2}\kappa_0\rho_2) \sin \theta d\theta \leq 0 \tag{9}$$

with the function $\rho_2(u, \theta)$ being an analogue to the news functions of electromagnetic field squared $X^2 + Y^2$ (see (14) in [6]).

Since we admit spacetimes with only "local" \mathcal{J}^+ , we assume Eqs. (2)–(9) to be satisfied for $\phi \in \langle 0, 2\pi \rangle$, however not necessarily on the whole sphere, i.e., for all $\theta \in \langle 0, \pi \rangle$, but only in some open interval of θ .

Let us follow a similar procedure to that one used for the electrovacuum case and assume here again the spacetime to have another Killing vector η which forms a two-dimensional Lie algebra with the axial one, $\xi = \partial/\partial\phi$, i.e., we assume $[\eta, \xi] = 0$ (see the Lemma in Sec. 2 in [2]). Hence, the components of η^α are independent of ϕ .

We introduce the standard null tetrad $\{k^\alpha, m^\alpha, t^\alpha, \bar{t}^\alpha\}$ (for details see (11) and the paragraph above in [3]), with bar denoting the complex conjugation:

$$\begin{aligned}
k_\alpha &= [1, 0, 0, 0] , \quad m_\alpha = [\tfrac{1}{2}Vr^{-1}e^{2\beta}, e^{2\beta}, 0, 0] , \\
t_\alpha &= \tfrac{1}{2}r(\cosh 2\delta)^{-\frac{1}{2}} \left[(1 + \sinh 2\delta)e^\gamma U + \cosh 2\delta e^{-\gamma} W + i[(1 - \sinh 2\delta)e^\gamma U - \cosh 2\delta e^{-\gamma} W] , \right. \\
&\quad \left. 0, -(1 + \sinh 2\delta + i(1 - \sinh 2\delta))e^\gamma, -(1 - i) \cosh 2\delta \sin \theta e^{-\gamma} \right] ,
\end{aligned} \tag{10}$$

and decompose the additional Killing vector η^α in this null tetrad

$$\eta^\alpha = Ak^\alpha + Bm^\alpha + \tilde{f}(t_R^\alpha + t_I^\alpha) + \tilde{g}(t_R^\alpha - t_I^\alpha) , \tag{11}$$

where $A, B, \tilde{f}, \tilde{g}$ are general functions of u, r, θ and $t^\alpha = t_R^\alpha + it_I^\alpha$.

The Killing vector η^α has to satisfy the Killing equations (all of them are written down in [14])

$$\mathcal{L}_\eta g_{\alpha\beta} = 0 . \tag{12}$$

The easiest one among them is the equation

$$\mathcal{L}_\eta g_{11} = 2e^{2\beta}B_{,r} = 0 , \tag{13}$$

which implies

$$B = B(u, \theta) . \tag{14}$$

We solve the other Killing equations asymptotically assuming that the coefficients A , \tilde{f} , \tilde{g} can be expanded in powers of r^{-k} . Then equations $\mathcal{L}_\eta g_{22} = 0$, $\mathcal{L}_\eta g_{12} = 0$, $\mathcal{L}_\eta g_{13} = 0$ imply

$$\begin{aligned} A &= A^{(-1)}r + A^{(0)} + \frac{A^{(1)}}{r} + \mathcal{O}(r^{-2}) , \\ \tilde{f} &= f^{(-1)}r + f^{(0)} + \frac{f^{(1)}}{r} + \mathcal{O}(r^{-2}) , \\ \tilde{g} &= g^{(-1)}r + g^{(0)} + \frac{g^{(1)}}{r} + \mathcal{O}(r^{-2}) , \end{aligned} \tag{15}$$

where the coefficients $A^{(k)}$, $f^{(k)}$, $g^{(k)}$ are functions of u and θ .

Since the null dust field decays at infinity in the same way as the electromagnetic field in [3], it does not enter the Killing equations in the leading order in r^{-k} as in [3] (see Eqs. (19)–(25) therein) and their solution is thus identical to the solution in the electrovacuum case in [3] and even the solution obtained in the vacuum case examined in [2]:

$$\begin{aligned} A^{(-1)} &= k \cos \theta , \\ f^{(-1)} &= -k \sin \theta , \\ B &= -ku \cos \theta + \alpha(\theta) , \end{aligned} \tag{16}$$

where $k = \text{const}$ and α is an arbitrary function of θ and

$$g^{(-1)} = h \sin \theta , \tag{17}$$

where $h = \text{const}$. One can easily find (using Eqs. (10), (11) and (15)) that the contribution of h to the vector field η^α is just constant multiple of the axial Killing vector $\partial/\partial\phi$, $\eta^\phi = h + \mathcal{O}(r^{-1})$, and so we may, without loss of generality, put $h = 0$. Therefore, in the lowest order of r^{-1} the general asymptotic form of the Killing vector η turns out to be

$$\eta^\alpha = [-ku \cos \theta + \alpha(\theta) , kr \cos \theta + \mathcal{O}(r^0) , -k \sin \theta + \mathcal{O}(r^{-1}) , \mathcal{O}(r^{-1})] , \tag{18}$$

where k is a constant, α – an arbitrary function of θ . Thus, assuming the presence of a null dust field satisfying the boundary conditions (3)–(5) and Killing vectors which need not be hypersurface orthogonal, we arrive in the leading order of the asymptotic expansion at the same conclusion obtained in Ref. [2] in the vacuum case with hypersurface orthogonal $\partial/\partial\phi$ or in Ref. [3] for the electrovacuum case which Killing vectors are not hypersurface orthogonal. When $k = 0$, the vector field (18) generates supertranslations.

Assuming $k \neq 0$, one can find a Bondi-Sachs coordinate system with $\alpha = 0$ by making a supertranslation, as was shown in [3]. Hence, we put $\alpha = 0$ in Eq. (18) and without loss of generality we choose $k = 1$. Then $B = -u \cos \theta$, $A^{(-1)} = \cos \theta$, $f^{(-1)} = -\sin \theta$ and $g^{(-1)} = 0$ and the asymptotic form of the Killing vector field η is

$$\eta^\alpha = [-u \cos \theta , r \cos \theta + \mathcal{O}(r^0) , -\sin \theta + \mathcal{O}(r^{-1}) , \mathcal{O}(r^{-1})] , \tag{19}$$

that is the boost Killing vector. It generates the Lorentz transformations along the axis of axial symmetry.

The conclusion of this section is thus following:

Suppose that an axially symmetric spacetime with null dust admits a “piece” of \mathcal{J}^+ in the sense that the Bondi-Sachs coordinates can be introduced in which the metric takes the form (2), (6) and the asymptotic forms of the energy-mass density and the null vector field of the null dust is given by (3)–(5). If this spacetime admits an additional Killing vector forming with the axial Killing vector a two-dimensional Lie algebra, then the additional Killing vector has asymptotically the form (18). For $k = 0$ it generates a supertranslation; for $k \neq 0$ it generates a boost along the symmetry axis.

However in the next sections we see that the boost symmetry is in fact not allowable.

III. THE SUPERTRANSLATIONAL KILLING FIELD

In this section, assuming $k = 0$ for the Killing field (18), we consider the Killing equations in higher orders of r^{-1} and arrive at the same equations as (31)–(48) in [3] with the same solutions (49)–(55) therein:

$$\begin{aligned}
c &= \frac{1}{2}uB^{-1}(B_{,\theta\theta} - B_{,\theta}\cot\theta) , \quad d = d(\theta) , \quad M = -uc_{,u}^2 - B^{-1}[A^{(1)} + B_{,\theta}(c_{,\theta} + 2c\cot\theta)] , \\
A^{(0)} &= \frac{1}{2}(B_{,\theta\theta} + B_{,\theta}\cot\theta + B) , \quad f^{(0)} = f^{(0)}(\theta) = -B_{,\theta} , \quad g^{(0)} = 0 , \\
A^{(1)} &= A^{(1)}(\theta) , \quad f^{(1)} = B(c_{,\theta} + 2c\cot\theta) , \quad g^{(1)} = B(d_{,\theta} + 2d\cot\theta) - B_{,\theta}d ,
\end{aligned} \tag{20}$$

where B , $A^{(1)}$ and d are arbitrary functions of θ . Thus, in the supertranslational case for the null dust the Weyl tensor is also non-radiative as it was in electrovacuum spacetimes. Substituting the metric functions (6) into the null tetrad (10) and coefficients (20), the expansion of the Killing vector reads

$$\begin{aligned}
\eta^\mu &= \left[B(\theta) , \frac{1}{2}(B_{,\theta\theta} + B_{,\theta}\cot\theta) + [-B_{,\theta\theta} - 2B_{,\theta}B_{,\theta\theta\theta} + 2B_{,\theta}^2B_{,\theta\theta}B^{-1} - 2B_{,\theta}^3\cot\theta B^{-1} \right. \\
&\quad \left. + B_{,\theta}^2(3\cot\theta^2 - 2\sin^{-2}\theta)]B^{-1}\frac{u}{4r} + \mathcal{O}(r^{-2}) , -B_{,\theta}\frac{1}{r} + B_{,\theta}\frac{c}{r^2} + \mathcal{O}(r^{-3}) , B_{,\theta}\frac{d}{r^2\sin\theta} + \mathcal{O}(r^{-3}) \right] .
\end{aligned} \tag{21}$$

Now let us turn to the asymptotic behaviour of the null dust. It is easy to show that if η is a Killing vector then the Lie derivative of the Riemann tensor with respect to this vector vanishes, and then also the Lie derivative of the Ricci tensor vanishes. And if, in addition, the Ricci scalar is zero and Einstein's equations are satisfied, then the following equations hold

$$\mathcal{L}_\eta T_{\mu\nu} = 0 . \tag{22}$$

Substituting η from (21), equations (22) get in the leading order the form

$$L_\eta T_{00} = 0 \quad (r^{-2}) : \quad \rho_{2,u}B = 0 , \tag{23}$$

$$L_\eta T_{01} = 0 \quad (r^{-4}) : \quad \rho_{2,u}\mathcal{U}B + \rho_2(\mathcal{U}_{,u}B - \mathcal{T}B_{,\theta}) = 0 , \tag{24}$$

$$L_\eta T_{02} = 0 \quad (r^{-2}) : \quad \rho_{2,u}\mathcal{T}B + \rho_2(\mathcal{T}_{,u}B - B_{,\theta}) = 0 , \tag{25}$$

$$L_\eta T_{03} = 0 \quad (r^{-2}) : \quad \rho_{2,u}\mathcal{F}B + \rho_2\mathcal{F}_{,u}B = 0 , \tag{26}$$

$$L_\eta T_{11} = 0 \quad (r^{-6}) : \mathcal{U}[\rho_{2,u}\mathcal{U}B + 2\rho_2(\mathcal{U}_{,u}B - \mathcal{T}B_{,\theta})] = 0 , \tag{27}$$

$$L_\eta T_{12} = 0 \quad (r^{-4}) : \quad \rho_{2,u}\mathcal{U}\mathcal{T}B + \rho_2[(\mathcal{U}\mathcal{T})_{,u}B - (\mathcal{U} + \mathcal{T}^2)B_{,\theta}] = 0 , \tag{28}$$

$$L_\eta T_{13} = 0 \quad (r^{-4}) : \quad \rho_{2,u}\mathcal{U}\mathcal{F}B + \rho_2[(\mathcal{U}\mathcal{F})_{,u}B - \mathcal{T}\mathcal{F}B_{,\theta}] = 0 , \tag{29}$$

$$L_\eta T_{22} = 0 \quad (r^{-2}) : \mathcal{T}[\rho_{2,u}\mathcal{T}B + 2\rho_2(\mathcal{T}_{,u}B - B_{,\theta})] = 0 , \tag{30}$$

$$L_\eta T_{23} = 0 \quad (r^{-2}) : \quad \rho_{2,u}\mathcal{T}\mathcal{F}B + \rho_2[(\mathcal{T}\mathcal{F})_{,u}B - \mathcal{F}B_{,\theta}] = 0 , \tag{31}$$

$$L_\eta T_{33} = 0 \quad (r^{-2}) : \quad \rho_{2,u}\mathcal{F}B + 2\rho_2\mathcal{F}_{,u}B = 0 . \tag{32}$$

If $\rho_2 = 0$, then all equations are trivially satisfied and it can be shown that we deal with a vacuum spacetime with an arbitrary supertranslational symmetry. If we suppose B and ρ_2 to be non-vanishing, then the first equation (23) implies

$$\rho_2 = \rho_2(\theta) , \tag{33}$$

and from Eqs. (26) or (32), (25) or (30), and (24), (27) or (29) it follows

$$\mathcal{F}_{,u} = 0 , \tag{34}$$

$$\mathcal{T}_{,u} = B_{,\theta}B^{-1} , \tag{35}$$

$$\mathcal{U}_{,u} = B_{,\theta}B^{-1}\mathcal{T} , \tag{36}$$

which yield

$$\mathcal{F} = \mathcal{F}_0(\theta) , \tag{37}$$

$$\mathcal{T} = B_{,\theta}B^{-1}u + \mathcal{T}_0(\theta) , \tag{38}$$

$$\mathcal{U} = \frac{1}{2}B_{,\theta}^2B^{-2}u^2 + B_{,\theta}B^{-1}\mathcal{T}_0u + \frac{1}{2}(\mathcal{T}_0^2 + \mathcal{F}_0^2\sin^2\theta) , \tag{39}$$

where we used condition (5). Comparing (8) with (20) we obtain the following equation determining the function $B(\theta)$ for the given “null dust” news function $\rho_2(\theta)$:

$$\left\{ \frac{B^2}{\sin\theta} \left[\frac{\sin^3\theta}{2B} \left(\frac{B_{,\theta}}{\sin\theta} \right)_{,\theta} \right]_{,\theta} \right\}_{,\theta} = \kappa_0 \rho_2 B^2 \sin\theta . \tag{40}$$

If a spacetime admits a global null infinity this case with a non-zero ρ_2 is not very physical since (9) then implies permanent linear decreasing of the total Bondi mass m .

IV. THE BOOST KILLING FIELD

In this section we investigate the boost case, $k = 1$ and $\alpha = 0$, similarly as the previous one by expanding the Killing equations in higher orders of r^{-1} . We obtain the same conditions for the coefficients $A^{(k)}$, $f^{(k)}$ and $g^{(k)}$ as Eqs. (81)–(89) and (104)–(112) in [3] with the identical solutions

$$A^{(0)}(u, \theta) = \frac{1}{2}u \cos \theta , \quad (41)$$

$$f^{(0)}(u, \theta) = -u^2 w + \mathcal{K}(w) = -u \sin \theta + \mathcal{K}(\sin \theta / u) , \quad (42)$$

$$g^{(0)}(u, \theta) = g^{(0)}(w) , \quad (43)$$

$$A^{(1)}(u, \theta) = \frac{\cos \theta}{w} \left(\frac{1}{2} \mathcal{K}_{,ww} w^2 + 2 \mathcal{K}_{,w} w + \mathcal{K} \right) - \frac{\cos \theta}{8 \sin^2 \theta} \left(4 \mathcal{K}^2 + g^{(0)2} \right) + \frac{\cos \theta}{u^2} \mathcal{L}(w) , \quad (44)$$

$$f^{(1)}(u, \theta) = \frac{1}{4 \sin \theta} (-2 \mathcal{K}^2 + g^{(0)2}) + u \cot^2 \theta (\mathcal{K}_{,w} w + \mathcal{K}) , \quad (45)$$

$$g^{(1)}(u, \theta) = \frac{1}{2} u g^{(0)} - \frac{g^{(0)} \mathcal{K}}{\sin \theta} + \frac{1}{2} u \cot^2 \theta (g^{(0)}_{,w} w + g^{(0)}) , \quad (46)$$

$$c(u, \theta) = -\frac{\mathcal{K}(w)}{uw} \rightarrow c_{,u}(u, \theta) = \frac{\mathcal{K}(w)_{,w}}{u^2} , \quad (47)$$

$$d(u, \theta) = -\frac{g^{(0)}(w)}{2 \sin \theta} \rightarrow d_{,u}(u, \theta) = \frac{g^{(0)}(w)_{,w}}{2u^2} , \quad (48)$$

$$M(u, \theta) = \frac{1}{2u} (\mathcal{K}_{,ww} w + 2 \mathcal{K}_{,w}) + \frac{\mathcal{L}(w)}{u^3} = \frac{1}{2 \sin \theta} (w^2 \mathcal{K}_{,w})_{,w} + \frac{\mathcal{L}}{u^3} , \quad (49)$$

where $\mathcal{K}(w)$ and $g^{(0)}(w)$ are arbitrary functions of w

$$w = \frac{\sin \theta}{u} , \quad (50)$$

and the integration function $\mathcal{L}(w)$ entering the expression for the mass aspect has the form

$$\mathcal{L}(w) = \frac{\lambda(w)}{w^3} , \quad (51)$$

with λ satisfying the equation

$$\lambda(w)_{,w} = w^2 (\mathcal{K}_{,w}^2 + \frac{1}{4} g^{(0)2}_{,w} + \frac{1}{2} \kappa_0 \omega) - \frac{1}{2w} (w^3 \mathcal{K}_{,ww})_{,w} . \quad (52)$$

Here, \mathcal{K} and $g^{(0)}$ determine the gravitational news functions, $c_{,u}$ and $d_{,u}$, by the relations (47) and (48), ω determines the null dust news function, ρ_2 , given below by the relation (65). Hence, solving the last equation for λ for *given* \mathcal{K} , $g^{(0)}$, and ω , we find $\mathcal{L}(w)$ and thus the mass aspect $M(u, \theta)$ in the form of Eq. (49). The total mass m at \mathcal{J}^+ is then given by integrating Eq. (49) over the sphere:

$$m(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta d\theta = \frac{1}{4} \int_0^\pi (w^2 \mathcal{K}_{,w})_{,w} d\theta + \frac{1}{2} \int_0^\pi \frac{w \mathcal{L}}{u^2} d\theta . \quad (53)$$

Substituting the expansions of the metric functions, Eq. (6) into the null tetrad, Eq. (10), and coefficients A , B , \tilde{f} and \tilde{g} , Eqs. (41)–(49), into Eq. (11), we find the asymptotic form of the boost Killing vector to be

$$\eta^\mu = \left[-u \cos \theta , r \cos \theta + u \cos \theta + \cos \theta \left(\mathcal{K}_{,w} + \frac{\mathcal{K}}{w} \right) \frac{1}{r} + \mathcal{O}(r^{-2}) , \right. \\ \left. - \sin \theta - u \sin \theta \frac{1}{r} + u c \sin \theta \frac{1}{r^2} + \mathcal{O}(r^{-3}) , \quad u d \frac{1}{r^2} + \mathcal{O}(r^{-3}) \right] . \quad (54)$$

Finally, let us turn our attention to the asymptotic properties of the null dust represented by the energy-momentum tensor $T_{\mu\nu}$ which, as was shown in the previous section, has to have the vanishing Lie derivative (22) with respect to the Killing vector η^α , (54). Regarding (41)–(49), these equations in the first orders look as follows:

$$\mathcal{L}_\eta T_{00} = 0 \quad (r^{-2}) : -\cos\theta(u\rho_{2,u} + \tan\theta \rho_{2,\theta} - 4\rho_2) = 0, \quad (55)$$

$$\mathcal{L}_\eta T_{01} = 0 \quad (r^{-4}) : -\cos\theta[(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{U} + \rho_2(u\mathcal{U}_{,u} + \tan\theta \mathcal{U}_{,\theta} + 4\mathcal{U} + u\tan\theta \mathcal{T})] = 0, \quad (56)$$

$$\mathcal{L}_\eta T_{02} = 0 \quad (r^{-2}) : \cos\theta[(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{T} + \rho_2(u\mathcal{T}_{,u} + \tan\theta \mathcal{T}_{,\theta} + 4\mathcal{T} + u\tan\theta)] = 0, \quad (57)$$

$$\mathcal{L}_\eta T_{03} = 0 \quad (r^{-2}) : \cos\theta[(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{F} + \rho_2(u\mathcal{F}_{,u} + \tan\theta \mathcal{F}_{,\theta} + 5\mathcal{F})] = 0, \quad (58)$$

$$\mathcal{L}_\eta T_{11} = 0 \quad (r^{-6}) : -\cos\theta[(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{U} + 2\rho_2(u\mathcal{U}_{,u} + \tan\theta \mathcal{U}_{,\theta} + 2\mathcal{U} + u\tan\theta \mathcal{T})] = 0, \quad (59)$$

$$\mathcal{L}_\eta T_{12} = 0 \quad (r^{-4}) : \cos\theta\{(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{U}\mathcal{T} + \rho_2[u(\mathcal{U}\mathcal{T})_{,u} + \tan\theta (\mathcal{U}\mathcal{T})_{,\theta} + 4\mathcal{U}\mathcal{T} + u\tan\theta (\mathcal{U} + \mathcal{T}^2)]\} = 0, \quad (60)$$

$$\mathcal{L}_\eta T_{13} = 0 \quad (r^{-4}) : \cos\theta\{(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{U}\mathcal{F} + \rho_2[u(\mathcal{U}\mathcal{F})_{,u} + \tan\theta (\mathcal{U}\mathcal{F})_{,\theta} + 5\mathcal{U}\mathcal{F} + u\tan\theta \mathcal{T}\mathcal{F}]\} = 0, \quad (61)$$

$$\mathcal{L}_\eta T_{22} = 0 \quad (r^{-2}) : -\cos\theta[(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{T} + 2\rho_2(u\mathcal{T}_{,u} + \tan\theta \mathcal{T}_{,\theta} + 2\mathcal{T} + u\tan\theta)] = 0, \quad (62)$$

$$\mathcal{L}_\eta T_{23} = 0 \quad (r^{-2}) : -\cos\theta\{(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{T}\mathcal{F} + \rho_2[u(\mathcal{T}\mathcal{F})_{,u} + \tan\theta (\mathcal{T}\mathcal{F})_{,\theta} + 5\mathcal{T}\mathcal{F} + u\tan\theta \mathcal{F}]\} = 0, \quad (63)$$

$$\mathcal{L}_\eta T_{33} = 0 \quad (r^{-2}) : -\cos\theta[(u\rho_{2,u} + \tan\theta \rho_{2,\theta})\mathcal{F} + 2\rho_2(u\mathcal{F}_{,u} + \tan\theta \mathcal{F}_{,\theta} + 3\mathcal{F})] = 0. \quad (64)$$

The trivial solution is again $\rho_2 = 0$ which implies a vacuum boost-rotation symmetric spacetime. Let us assume $\rho_2 \neq 0$. Using variable w given by (50), Eq. (55) can be solved to yield

$$\rho_2 = \frac{\omega(w)}{u^2}, \quad (65)$$

with an arbitrary function $\omega(w)$. The sum of Eqs. (58) and (64) gives

$$\mathcal{F} = \frac{\mathcal{F}_0(w)}{u}, \quad (66)$$

($\mathcal{F}_0(w)$ being an arbitrary function) and their difference is then identically zero. Similarly, summing Eqs. (57) and (62), we obtain the equation for \mathcal{T} ,

$$\mathcal{T}_{,u} = -\tan\theta = -\frac{uw}{\sqrt{1-u^2w^2}}, \quad (67)$$

which leads to the solution

$$\mathcal{T} = \frac{\sqrt{1-u^2w^2}}{w} + \mathcal{T}_0(w) = u \cot\theta + \mathcal{T}_0(w), \quad (68)$$

where $\mathcal{T}_0(w)$ is an arbitrary integration function. Next, the difference of Eqs. (57) and (62) identically vanishes. We repeat the procedure for the sum of Eqs. (56) and (59) and find the solution

$$\mathcal{U} = -\frac{1}{2}u^2 + \mathcal{T}_0(w)\frac{\sqrt{1-u^2w^2}}{w} + \mathcal{U}_0(w) = -\frac{1}{2}u^2 + u \cot\theta \mathcal{T}_0(w) + \mathcal{U}_0(w), \quad (69)$$

($\mathcal{U}_0(w)$ is an arbitrary function of w). Then their difference is identically zero and also Eqs. (60), (61), (63) are identically satisfied.

The coefficients \mathcal{U} , \mathcal{T} and \mathcal{F} have, in addition, to fulfil the condition for the null vector (5). This, however, is in contradiction with the boost-rotation symmetric solutions (66), (68) and (69). Consequently, there are no asymptotically flat boost-rotation symmetric solutions of the Einstein equations with null dust. And the final conclusion reads:

Theorem

Suppose that an axially symmetric spacetime with null dust admits a “piece” of \mathcal{J}^+ in the sense that the Bondi-Sachs coordinates can be introduced in which the metric takes the form (2), (6) and the asymptotic forms of the radiation density and the null vector field of the null dust are given by (3)–(5). If this spacetime admits an additional Killing vector forming a two-dimensional Lie algebra with the axial Killing vector, then the additional Killing vector which has asymptotically the form (21) generates asymptotically supertranslations and the Weyl tensor is non-radiative (although one of the “gravitational” news functions, $c_{,u}$, is non-vanishing, however, it is a function only of θ as the “null dust” news function, ρ_2).

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